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Wallman Compactification and Matrices of Zeroes and Ones*

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Let M be an m by n matrix (where m and n are any finite or infinite cardinals) such that the entries of M are 0's or 1's and M contains the zero row $\bar{0}$ and the rows of M are closed under coordinatewise multiplication. We prove that M can be extended to an m by $n' \geqslant n$ matrix M' such that the entries of M' are 0's or 1's and M' contains the zero row $\bar{0}'$ and the extension preserves the zero products. Moreover, the newly introduced columns (if any) are pairwise distinct. Furthermore, if E' is any set of rows of M' with the property that for every finite subset of rows r_i' of E' there exists $j < n'$ such that $r_{ij}' = 1$, then every subset of rows of E' has the same property. We rephrase this by saying that if E' has the finite intersection property then E' has a nonempty intersection. We also show (this time by Zorn's lemma) that there exists an extension of M with all the abovementioned properties such that the extension preserves products sums, complements and the newly introduced columns (if any) are pairwise distinct in a stricter sense. In effect, our result shows that the classical Wallman compactification theorem can be formulated purely combinatorially requiring no introduction of any topology on n.

In what follows 0 and 1 stand for the zero and the unit real numbers respectively. Moreover, all the matrices have 0 or 1 as their entries and the dimensions of matrices are finite or infinite cardinal numbers.

Let M be an m by n matrix (where, as mentioned, m and n are any finite or infinite cardinals). Thus, for every $i < m$ the i-th row r_i of M is a dyadic sequence $(r_{ij})_{j<n}$ of type n.

In the sequel any statement which is made in connection with the multiplication (or product) of the rows of a matrix, refers to the coordinatewise multiplication of these rows.

As usual, a set S of rows of a matrix is called a *multiplicative system* if and only if S is closed under (coordinatewise) multiplication of its rows.

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Clearly, the set of all the finite products of a set E of rows of a matrix is a multiplicative system and (for obvious reasons) is called *the multiplicative system generated by* E.

As expected, a row of a matrix M is called the *zero row* of M if and only if every entry of that row is 0 (i.e., the zero row is a zero dyadic sequence).

Let E' be a set of rows of an m by n' matrix M'. As expected, we say that E' has a *nonempty intersection* if and only if there exists $j < n'$ such that

$$r'_{ij} = 1 \qquad \textit{for every row } r'_{ij} \in E'. \tag{1}$$

Also, as usual, we say that E' has the *finite intersection property* if and only if every finite subset of E' has a nonempty intersection. From this and (1), we have immediately:

LEMMA. *Let M' be a matrix and E' be a set of rows of M'. Then E' has the finite intersection property if and only if the zero row of M' is not an element of the multiplicative system generated by E'.*

We say that an m by n' matrix M' is an extension of an m by n matrix M if and only if for every $i < m$ the row r'_i of M' is an extension (in the sense of extension of a dyadic sequence) of the row r_i of M.

Based on the above, we prove:

THEOREM 1. *Let M be an m by n matrix whose rows are pairwise distinct, include the zero row $\bar{0}$ and are closed under multiplication. Then M can be extended to an m by n' matrix M' via extending every row r_i of M to a row r'_i of M' such that*:

(i) *The rows of M' include the zero row $\bar{0}'$.*

(ii) *The correspondence $r_i \to r'_i$ is one-to-one and preserves the zero and the zero-products, i.e., $(\bar{0})' = \bar{0}'$ and for any finitely many rows $r_u, r_v, \ldots, r_w$ of M if $r_u r_v \cdots r_w = \bar{0}$ then $r'_u r'_v \cdots r'_w = \bar{0}'$.*

(iii) *The columns of the matrix $M' - M$ are pairwise distinct.*

(iv) *If a set E' of rows of M' has the finite intersection property then E' has a nonempty intersection.*

Proof. Let us observe that the set R_j of rows r_i of M given by:

$$R_j = \{r_i \mid i < m \textit{ and } r_{ij} = 1\} \qquad \textit{for every} \quad j < n \tag{2}$$

is a multiplicative system which excludes $\bar{0}$, i.e., $\bar{0} \notin R_j$.

Let

$$\{R_v \mid v \in V\} \qquad \textit{with} \quad R_v \neq R_w \textit{ if } v \neq w \textit{ for every } v, w \in V \tag{3}$$

be the set of *all* the multiplicative systems R_v of rows of M each of which excludes $\bar{0}$ and such that $R_v \neq R_j$ for every $v \in V$ and every $j < n$.

Let us consider the cardinal number n' given by:

$$n' = n + \overline{\overline{V}} \tag{4}$$

Clearly, $n' \geqslant n$ and from (2), (3), (4), with an obvious renaming of the elements of V, it follows that:

$$\{R_j \mid j < n'\} \tag{5}$$

is the set of *all* the multiplicative systems R_j of rows of M each of which excludes $\bar{0}$.

To every row r_i of M let us correspond a dyadic sequence r_i' of type n' defined as:

$$\begin{aligned} r'_{ij} &= r_{ij} \qquad \textit{for } j < n \\ r'_{ij} &= 1 \qquad \textit{for } n \leqslant j < n' \quad \textit{if and only if} \quad r_i \in R_j . \end{aligned} \tag{6}$$

Obviously, r_i' is an extension of r_i. Let M' be a matrix whose rows are r_i' with $i < m$. Clearly, M' is an extension of M which is obtained via extending r_i to r_i'.

From (3), (4), (6) we have:

$$R_j = \{r_i \mid i < m \textit{ and } r'_{ij} = 1\} \qquad \textit{with} \qquad n \leqslant j < n'$$

which by (2), (6) implies:

$$R_j = \{r_i \mid i < m \textit{ and } r'_{ij} = 1\} \qquad \textit{for every } j < n'. \tag{7}$$

Since $\bar{0} \notin R_j$ for every $j < n'$, from (6) it follows that the extension $(\bar{0})'$ of $\bar{0}$ is the zero row $\bar{0}'$ of M'. Thus, (i) is established.

Since the rows of M are pairwise distinct, it follows that the correspondence $r_i \to r_i'$ is obviously one-to-one.

Now, let $r_u, r_v, \ldots, r_w$ be any finite number of rows of M such that $r_u r_v \cdots r_w = \bar{0}$. Let $j < n'$. Clearly, at least one of $r_u, r_v, \ldots, r_w$, say, r_v is not an element of R_j (since R_j is a multiplicative system of rows of M such that $\bar{0} \notin R_j$). But then, from (6) it follows that $r'_{vj} = 0$. Hence, for every $j < n'$ at least one of $r'_{uj}, r'_{vj}, \ldots, r'_{wj}$ is equal to 0 and consequently, for every $j < n'$ it is the case that $r'_{uj} r'_{vj} \cdots r'_{wj} = 0$ which implies that $r'_u r'_v \cdots r'_w = \bar{0}'$.

Thus, (ii) is established.

Next, from (3) it follows that the multiplicative systems R_j with $n \leqslant j < n'$ are pairwise distinct. Let $R_j \neq R_k$ with $n \leqslant j$, $k < n'$. Therefore, there exists a row r_i of M such that (without loss of generality), say, $r_i \in R_j$ and $r_i \notin R_k$.

But then, from (6), it follows that $r'_{ij} = 1$ and $r'_{ik} = 0$ which implies that the columns c_j and c_k of M' are distinct.

Thus, (iii) is also established.

Next, let E' be a set of rows of M' such that E' has the finite intersection property. Hence, by the Lemma, the multiplicative system $\{r'_i \mid i \in P\}$ generated by E' excludes $\bar{0}'$. Obviously,

$$E' \subseteq \{r'_i \mid i \in P\}. \tag{8}$$

However, $\{r_i \mid i \in P\}$ is a multiplicative system of rows of M which excludes $\bar{0}$. Because, if $\bar{0} = r_u r_v \cdots r_w$ with $u, v,..., w \in P$ then from (ii) it follows that $r'_u r'_v v \cdots r'_w = \bar{0}'$ contradicting that $\bar{0} \notin \{r'_i \mid i \in P\}$. But then from (5) it follows that:

$$\{r_i \mid i \in P\} = R_j \qquad \textit{for some} \quad j < n'$$

which, by (6) and (8) implies

$$r'_{ij} = 1 \qquad \textit{for every } r'_i \in E'$$

and which, in turn, by (1) implies that E' has a nonzero intersection.

Thus, (iv) is also established.

As usual, the *sum* of the same type of dyadic sequences r_i and r_h is defined to be the dyadic sequence r_k of the same type, where:

$$r_{kj} = 1 \qquad \textit{if and only if} \quad r_{ij} = 1 \quad \textit{or} \quad r_{hj} = 1$$

The sum of r_i and r_h is denoted by $r_i \dotplus r_h$.

As expected, the dyadic sequence which is obtained by exchanging the 0's and 1's in a dyadic sequence r_i is called the *complement* of r_i and is denoted by $C(r_i)$.

We say that the columns c_t of a matrix are pairwise distinct in the *strict sense* if and only if (using the obvious notation) $c_j \neq c_k$ implies

$$r_{ij} = r_{hk} = 0 \qquad \textit{and} \qquad r_{ik} = r_{hj} = 1 \qquad \textit{for somr } i \textit{ and } h$$

Based on the above, we prove:

THEOREM 2. *Let M be an m by n matrix whose rows are pairwise distinct, include the zero row $\bar{0}$ and are closed under multiplication. Then M can be extended to an m by n^* matrix M^* via extending every row r_i of M to a row r_i^* of M^* such that:*

(i)* *The rows of M^* include the zero row $\bar{0}^*$.*

(ii)* *The correspondence $r_i \to r_i^*$ is one to one and preserves the zero, the products, the sums and the complements, i.e., $(\bar{0})^* = \bar{0}^*$ and $(r_i r_h)^* = r_i^* r_h^*$ for* *every* $i, h < m$ *and* $(r_i \dotplus \cdots \dotplus r_h)^* = r_i^* \dotplus \cdots \dotplus r_h^*$ *for every* $i, h < m$ *and* $(C(r_i))^* = C(r_i^*)$ *for every* $i < m$.

(iii)* *The rows of* M^* *are closed under multiplication.*

(iv)* *The columns of the matrix* $M^* - M$ *are distinct in the strict sense.*

(v)* *If a set* E^* *of rows of* M^* *has the finite intersection property then* E^* *has a nonempty intersection.*

Proof. Let us remark that if R is a multiplicative system of rows of M such that $\bar{0} \notin R$ then by Zorn's lemma there exists a maximal (with respect to the property of excluding $\bar{0}$) multiplicative system R^* of rows of M such that $R \subseteq R^*$ and $\bar{0} \notin R^*$.

To prove the Theorem, we proceed as in the case of the proof of Theorem 1 with the only difference that we replace $\{R_v \mid v \in V\}$ given in (3) by the set $\{R_v^* \mid v \in V^*\}$ of all the maximal with respect to the property of excluding $\bar{0}$ multiplicative systems R_v of rows of M such that $R_v \neq R_j$ for every $v \in V^*$ and every $j < n$. Also, we replace n' given in (4) by n^* where $n^* = n + \bar{V}^*$.

But then, (i)* and the first part of (ii)* are proved precisely as (i) and (ii). To prove the second part of (ii)*, in view of (6) it is enough to show that for every i, $h < m$ and $n \leqslant j < n^*$ we have:

$$(r_i r_h) \in R_j^* \qquad \text{if and only if} \qquad r_i \in R_j^* \quad \text{and} \quad r_h \in R_j^*.$$

The "if" part follows trivially from the fact that R_j^* is a multiplicative system. To prove the "only if" part, let us assume to the contrary that $(r_i r_h) \in R_j^*$ and $r_i \notin R_j^*$. However, since R_j^* is a multiplicative system maximal with respect to the property of excluding $\bar{0}$, we see that $r_t r_i = \bar{0}$ for some $r_t \in R_j^*$ but then $r_t r_i r_h = \bar{0}$ implying that $\bar{0} \in R_j^*$ which is a contradiction. Thus, the second part of (ii)* is proved. To prove the third part of (ii)*, it suffices to prove the statement for the sum of two rows since the proof for the sum of any finite number of rows can be given in a similar way. For this purpose, in view of (6), it is enough to show that for every i, $h < m$ and $n \leqslant j < n^*$ we have:

$$(r_i \dotplus r_h) \in R_j^* \qquad \text{if and only if} \qquad r_i \in R_j^* \quad \text{or} \quad r_h \in R_j^*.$$

To prove the "if" part, let, say, $r_i \in R_j^*$ and assume to the contrary that $(r_i \dotplus r_h) \notin R_j^*$. However, since R_j^* is a multiplicative system maximal with respect to the property of excluding $\bar{0}$, we see that $(r_i \dotplus r_h)\, r_t = r_i r_t \dotplus r_h r_t = \bar{0}$ for some $r_t \in R_j^*$. But then obviously, $r_i r_t = r_h r_t = \bar{0}$ which, in view of the fact that $r_i r_t \in R_j^*$, implies $\bar{0} \in R_j^*$, which is a contradiction. To prove the "only if" part, let $(r_i \dotplus r_h) \in R_j^*$ and assume to the contrary that $r_i \notin R_j^*$ and $r_h \notin R_j^*$. But then again, $r_i r_p = \bar{0}$ and $r_h r_q = \bar{0}$ for some $r_p \in R_j^*$ and $r_q \in R_j^*$. Consequently, $r_i t_p r_q = r_h r_p r_q = \bar{0}$ which implies $(r_i \dotplus r_h)\, r_p r_q = \bar{0}$ which, in view of the fact that $(r_i \dotplus r_h)\, r_p r_q \in R_j^*$, again implies $\bar{0} \in R_j^*$, which is a contradiction. Thus, the third part of (ii)* is proved. To prove the fourth part of (ii)*, in view of (6), it is enough to show that for every $i < m$ and $n \leqslant j < n^*$ we have:

$$r_i \in R_j^* \qquad \textit{if and only if} \qquad C(r_i) \notin R_j^*.$$

Since $r_i C(r_i) = \bar{0}$ and since R_i^* is a multiplicative system of rows of M such that $\bar{0} \notin R_j^*$ we see that $r_i \in R_j^*$ and $C(r_i) \in R_j^*$ is impossible. Thus, it remains to show that $r_i \notin R_j^*$ and $C(r_i) \notin R_j^*$ is also impossible. But this is indeed the case since R_i^* being a multiplicative system maximal with respect to the property of excluding $\bar{0}$, would otherwise imply $r_i r_p = \bar{0}$ for some $r_p \in R_j^*$ and $C(r_i)\, r_q = \bar{0}$ for some $r_q \in R_j^*$. Consequently, $r_i r_p r_q = C(r_i)\, r_p r_q = \bar{0}$ which would imply $r_p r_q = \bar{0}$, contradicting that $\bar{0} \notin R_j^*$. Thus, the fourth part of (ii)* is also proved.

Clearly, the second part of (ii)*, in view of the fact that the rows of M are closed under multiplication, implies (iii)*. We observe also that (iv)* follows readily from the maximality of R_j^* and R_k^* with $n \leqslant j,\ k < n^*$ since in this case if $R_j^* \neq R_k^*$ then $(R_k^* - R_j^*) \neq \phi \neq (R_j^* - R_k^*)$.

The proof of (v)* is like that of (iv) since every multiplicative system $\{R_i \mid i \in P\}$ which excludes $\bar{0}$ is contained (in view of our abovementioned remark concerning the implication of Zorn's lemma) in a maximal with respect to the property of excluding $\bar{0}$) multiplicative system R_j^* of M for some $j < n^*$.

Thus, the Theorem is proved.

Remark. Let n be a topological space. Clearly, the characteristic functions of all the closed sets of n can be arranged to form the rows of an m by n matrix M which has all the properties mentioned in Theorem 1. Let us topologize the cardinal n' given in (4) such that the rows r_i' of matrix M' are the characteristic functions of subbasic closed sets of n'. But then, Alexander s subbase lemma, in view of (iv), implies that n' is a compact space containing n as a subspace. If the topology on n is T_0 then (iii) and (iv) imply that n' is a compact T_0 space.

On the other hand, based on Theorem 2 it can be readily shown that if we start with a topological space n then (in view of (ii)* and (iii)*) the corresponding subbasic closed sets of n^* are actually basic closed sets of n^*. But then (ii)* implies that n is dense in n^* and hence (v)* implies that n^* is a compactification of n. If the topology on n is T_1 then, in view of (iv)*, we see that n^* is a T_1 compactification of n which is indeed the Wallman compactification [2, p. 139] of n. As expected, in this case for every $j < n$ the multiplicative system R_j given by (2) is also a maximal with respect to the property of excluding $\bar{0}$ multiplicative system of M. This follows readily from (2) and the fact that every one-element subset of a T_1 topological space is a closed set of that space.

This paper is a version of [1] where compactness was not defined via the finite intersection property.

References

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